A queueing model with start-up/close-down times and retrial customers

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Abstract

We consider a queueing system with Poisson arrivals and arbitrarily distributed service times, vacation times, start up and close down times. The model accepts two types of customers, the ordinary and the retrial customers and the server takes a single vacation each time he becomes free. For such a model the stability conditions are investigated and the system state probabilities are obtained both in a transient and in a steady state and used to derive some important measures of the system performance.

Keywords: Poisson arrivals, Start up times, close down times, vacation, retrial queue, general services.

1 Introduction

Queueing models with vacation periods and start up/close down times have been proved very useful to model telecommunication systems and many other queueing situations containing "mechanical parts" that need a "preparation" (start up) before use and a switch off-maintenance period after use (computer systems, manufacturing systems e.t.c.). Applications of such kind of queueing models in SVC - based virtual LAN- emulation and in IP over ATM networks have been described in details in Sakai et al. [1], Niu & Takahashi [2] and in the references therein.

In all models, described and investigated above, the arriving "customers" are queued up and wait to be served. On the other hand, it is easy to realize that such kind of real situations accept, in many cases, and a second kind of "customers" that do not wait in a queue but instead, if they find upon arrival the server unavailable, they depart from the system and repeat their arrival later until succeed to be served. As a simple example of such a situation one can consider an X-ray unit or a tomographic unit, where the machine needs a special time to start working and to close down where there are no more patients waiting, while external phone calls of patients that ask for the results of their examinations or ask for medical advice, arrive and engage the "server ". Queueing systems with retrial customers are widely used in the literature to model telephone switching systems, telecommunication systems and computer networks. For a complete survey on past papers on such kind of models see Falin & Templeton [3], Kulkarni & Liang [4] and Artalejo [5].

In this paper, the two important features, i.e., the start up/close down feature and the retrial feature are combined together, for the first time in the literature. Thus here we study for the first time a queueing model of vacation-start up/close down nature accepting two types of customers, the ordinary customers that are queued up and wait for service and the retrial customers. The server needs a start up time before starts working on customers (different start up for each type of customers), a close down period upon finishing the job, and when he is free he departs for a single vacation. Moreover the ordinary customers have a kind of priority upon retrial customers, in the sense that the arrival of an ordinary customer interrupts the start up time of the retrial customer (if any), and the server starts to be prepared to serve the ordinary customer.

The article is organized as follows. A full description of the model and some, very useful for the analysis, preliminary results are given in section 2 and 3 respectively. The time dependent analysis of the system state probabilities is performed in section 4, while in section 5 the conditions for statistical equilibrium are investigated. Finally, the generating functions of the steady state probabilities are obtained in section 6 and used to obtain, in section 7, some important measures of the system performance.

2 The Model

Consider a single server queue accepting two types of customers. The P_1 customers (ordinary customers) arrive according to a *Poisson* distribution parameter λ_1 and queued up in an ordinary queue waiting to be served. The P_2 customers (retrial customers) arrive according to a *Poisson* distribution parameter λ_2 and, if find the server unavailable, they leave the system and join a retrial box from where they retry independently, after an exponential time parameter α, to find a position for service.

To start serving the P_1customers waiting in the queue or the P_2 customer who found a position for service, the server needs a start up period S_i, $i = 1, 2$ (different for each type of customers), distributed according to a general distribution with distribution function (D.F.) $S_i(x)$, probability density function (p.d.f.) $s_i(x)$ and finite mean value $\bar{s}_i$, $i = 1, 2$ respectively. Moreover the server, upon finishing all tasks in the queue and in the service area (the retrial box is not necessarily empty at this point), operates a close down period C arbitrarily distributed with D.F. $C(x)$, p.d.f. $c(x)$ and finite mean $\bar{c}$. During the close down period no retrial customer can access the service facility, while if a P_1 customer arrives during C, the server returns to the serving mode and starts serving P_1 customers but now with a different start up period S_3 with

D.F. $S_3(x)$, p.d.f. $s_3(x)$ and finite mean $\bar{s}_3$. This can be explained by the fact that after an incomplete close down period it is natural for the server to need a different start up time to transfer again the system in the serving mode.

When a close down period is successfully completed the server departs for a single vacation V which length is arbitrarily distributed with D.F. $V(x)$, p.d.f. $v(x)$, and finite mean $\bar{v}$. If, in the end of the vacation, there are P_1 customers waiting in the queue the server operates an S_1 period etc., while if the queue is empty he remains idle waiting for the first customer, from outside or from the retrial box, to start working again.

It is natural for the ordinary P_1 customers to have a kind of priority over the retrial P_2 customers. Thus, if a P_1 customer arrives during the start up time of a P_2 customer then this start up period is interrupted and a S_1 time followed by a busy period of P_1 customers and a close down period begins. The interrupted P_2 customer does not return to the retrial box but he restarts his start up time from the beginning when this close down period of P_1 customers is finished. On the other hand the arrival of a P_1 customer cannot interrupt the service time of a P_2 customer. In the later case, the service of the P_2 customer is completed and in the sequel the server starts working (start up plus busy period) on the P_1 customers.

Finally, the service times of both type of customers are arbitrarily distributed with D.F. $B_i(x)$, p.d.f. $b_i(x)$ and finite mean value $\bar{b}_i$ for $i = 1, 2$ respectively, while all random variables defined above are assumed to be independent.

3 Preliminary Results

We agree from here on to denote in general by $a^*(s)$ the Laplace Transform (L.T.) of any function $a(t)$. Let us denote now by $\mathcal{B}^{(i)}$ the duration of a busy period of P_1 customers which starts with $i = 1, 2, \ldots$ P_1 customers, and let $\mathcal{N}(\mathcal{B}^{(i)})$ be the number of P_2 customers arrive during $\mathcal{B}^{(i)}$. Define

$$g_m^{(i)}(t)dt = \Pr[t < \mathcal{B}^{(i)} \leq t + dt,\ \mathcal{N}(\mathcal{B}^{(i)}) = m].$$

Then it is known from Langaris & Katsaros [6] that

$$g^{*(i)}(s, z_2) = \int_0^\infty e^{-st} \sum_{m=0}^{\infty} g_m^{(i)}(t) z_2^m dt = x^i(s, z_2),$$

where $x(s, z_2)$ is the root in z_1 with the smallest absolute value of the equation

$$z_1 - b_1^*(s + \lambda_1(1 - z_1) + \lambda_2(1 - z_2)) = 0.$$

Let now denote by R the time interval from the beginning of a close down period until this period be successfully completed and let $N(R)$ be the number of P_2 customers arriving during R. If we define

$$r_j(t)dt = \Pr[t < R \leq t + dt,\ N(R) = j], \qquad r^*(s, z_2) = \int_0^\infty e^{-st} \sum_{j=0}^{\infty} r_j(t) z_2^j dt,$$

and denote

$$p_{im}(t) = e^{-\lambda_1 t}\frac{(\lambda_1 t)^i}{i!}e^{-\lambda_2 t}\frac{(\lambda_2 t)^m}{m!}, \qquad a(s, z_1, z_2) = s + \lambda_1(1 - z_1) + \lambda_2(1 - z_2),$$

then it is clear that

$$\begin{aligned} r_j(t) = \;& p_{0j}(t)c(t) + \{\lambda_1 \textstyle\sum_{k=0}^{j} p_{0k}(t)[1 - C(t)]\} \\ & *\{\textstyle\sum_{i=0}^{\infty}\sum_{m=0}^{j-k}\; p_{im}(t)\; s_3(t)\} * \sum_{l=0}^{j-k-m} g_l^{(i+1)}(t) * r_{j-k-m-l}(t), \end{aligned} \tag{1}$$

and so finally

$$r^*(s, z_2) = \frac{c^*(a(s, 0, z_2))}{1 - M(s, x(s, z_2), z_2)},$$

with

$$M(s, z_1, z_2) = \lambda_1 z_1 s_3^*(a(s, z_1, z_2)\;)\frac{1 - c^*(a(s, 0, z_2))}{a(s, 0, z_2)}. \tag{2}$$

Let now Q be the time interval from the beginning of a vacation period until the epoch at which the server becomes idle, and let $N(Q)$ the number of P_2 customers arrive during Q. If we define

$$q_j(t)dt = \Pr[t < Q \leq t + dt,\; N(Q) = j], \qquad q^*(s, z_2) = \int_0^{\infty} e^{-st}\sum_{j=0}^{\infty} q_j(t) z_2^j dt,$$

then

$$\begin{aligned} q_j(t) = \;& p_{0j}(t)v(t) + \textstyle\sum_{i=1}^{\infty}\sum_{k=0}^{j} p_{ik}(t)v(t) * \sum_{l=0}^{\infty}\sum_{m=0}^{j-k} p_{lm}(t)s_1(t) \\ & *\textstyle\sum_{r=0}^{j-k-m} g_r^{(i+l)}(t) * \sum_{n=0}^{j-k-m-r} r_n(t) * q_{j-n-k-m-r}(t), \end{aligned} \tag{3}$$

and so

$$q^*(s, z_2) = \frac{v^*(a(s, 0, z_2))}{1 - \;[v^*(a(s,\; z_1, z_2)) - v^*(a(s, 0, z_2)]s_1^*(a(s,\; z_1, z_2))r^*(s, z_2)}. \tag{4}$$

Note here that if we denote

$$e(s, z_1, z_2) = \frac{c^*(a(s,0,z_2))v^*(a(s,0,z_2))}{1-M(s,z_1,z_2)-c^*(a(s,0,z_2))[v^*(a(s,\; z_1,z_2))-v^*(a(s,0,z_2)]s_1^*(a(s,\; z_1,z_2))}, \tag{5}$$

then

$$q^*(s, z_2) = \frac{e(s, x(s, z_2), z_2)}{r^*(s, z_2)}. \tag{6}$$

If finally $\rho_v = E(N(Q))$ then by differentiating (4) with respect to z_2 we arrive at $\rho_v = \bar{\rho}_v/(1 - \lambda_1\bar{b}_1)$ where

$$\begin{aligned} \bar{\rho}_v \;=\; & \frac{\lambda_2}{\lambda_1 v^*(\lambda_1)c^*(\lambda_1)}\{(1 + \lambda_1\bar{s}_3)(1 - c^*(\lambda_1))(1 - v^*(\lambda_1)) \\ & +\lambda_1 c^*(\lambda_1)(\bar{v} + \bar{s}_1(1 - v^*(\lambda_1))\}. \end{aligned}$$

We are now ready to define the " service completion time" of a P_2 customer as the time $\bar{W}_2$ elapsed from the epoch at which this customer succeed to find a position for service until the time the server departs for a vacation. Let $N(\bar{W}_2)$ the number of new P_2 customers that arrive during $\bar{W}_2$ and

$$\bar{w}_j(t)dt = \Pr[t < \bar{W}_2 \leq t+dt,\ N(\bar{W}_2) = j], \qquad \bar{w}^*(s,z_2) = \int_0^\infty e^{-st}\sum_{j=0}^\infty \bar{w}_j(t)z_2^j dt,$$

Then by writing for $\bar{w}_j(t)$ a similar expression as in (1) and (3) and taking Laplace transforms we arrive easily at

$$\bar{w}^*(s,z_2) = \frac{L(s,x(s,z_2),z_2)r^*(s,z_2)}{1-K(s,x(s,z_2),z_2)r^*(s,z_2)}, \tag{7}$$

where

$$\begin{aligned} L(s,z_1,z_2) = \;& s_2^*(a(s,0,z_2))\{b_2^*(a(s,0,z_2)) + s_1^*(a(s,z_1,z_2))[b_2^*(a(s,z_1,z_2)) \\ & -b_2^*(a(s,0,z_2))]\}, \\ K(s,z_1,z_2) = \;& \lambda_1 z_1 s_1^*(a(s,z_1,z_2))\frac{1-s_2^*(a(s,0,z_2))}{a(s,0,z_2)}. \end{aligned} \tag{8}$$

If finally $\rho_c = E(N(\bar{W}_2))$ then by differentiating (7) with respect to z_2 we arrive at $\rho_c = \bar{\rho}_c/(1-\lambda_1\bar{b}_1)$

$$\begin{aligned} \bar{\rho}_c \;=\; & \frac{\lambda_2}{\lambda_1 s_2^*(\lambda_1)c^*(\lambda_1)}\{(1+\lambda_1\bar{s}_3)(1-c^*(\lambda_1)) \\ & +c^*(\lambda_1)[(1+\lambda_1\bar{s}_1)(1-s_2^*(\lambda_1)) + \lambda_1 s_2^*(\lambda_1)(\bar{b}_2 + \bar{s}_1(1-b_2^*(\lambda_1))]\}. \end{aligned}$$

The "generalized service completion time " of a P_2 customer, W_2 say, can be defined as the time elapsed from the epoch at which this customer succeed to find a position for service until the time the server is again idle and so free to accept the next customer (from outside or from the retrial box). Let $N(W_2)$ the number of new P_2 customers that arrive during W_2 and

$$w_j(t)dt = \Pr[t < W_2 \leq t+dt,\ N(W_2) = j], \qquad w^*(s,z_2) = \int_0^\infty e^{-st}\sum_{j=0}^\infty w_j(t)z_2^j dt,$$

then it is clear that

$$w^*(s,z_2) = \bar{w}^*(s,z_2)q^*(s,z_2),$$

and from (6), (7)

$$w^*(s,z_2) = \frac{L(s,x(s,z_2),z_2)e(s,x(s,z_2),z_2)}{1-K(s,x(s,z_2),z_2)r^*(s,z_2)}, \tag{9}$$

while, by suitable differentiations, the mean number of P_2 customers arriving during W_2 and the duration of W_2 are given by

$$\rho_w = E(N(W_2)) = \frac{\bar{\rho}_v + \bar{\rho}_c}{1-\lambda_1\bar{b}_1}, \qquad E(W_2) = \frac{1-\rho_w}{\lambda_2}. \tag{10}$$

We define finally the "generalized busy period" of P_1 customers as the time interval, W_1 say, from the epoch at which a P_1 customer arrives in an idle server until the epoch at which the server remains idle again. If as before $N(W_1)$ is the number of new P_2 customers arrive during W_1 and

$$d_j(t)dt = \Pr[t < W_1 \le t+dt,\ N(W_1) = j], \qquad d^*(s, z_2) = \int_0^\infty e^{-st} \sum_{j=0}^\infty d_j(t) z_2^j dt,$$

then

$$d_j(t) = \sum_{i=0}^\infty \sum_{m=0}^j p_{im}(t) s_1(t) * \sum_{l=0}^{j-m} g_l^{(i+1)}(t) * \sum_{k=0}^{j-m-l} r_k(t) * q_{j-m-l-k}(t),$$

and so

$$d^*(s, z_2) = x(s, z_2) s_1^*(a(s, x(s, z_2), z_2)) r^*(s, z_2) q^*(s, z_2). \tag{11}$$

If finally we differentiate (11), with respect to z_2 and s, we obtain

$$E(N(W_1)) = \bar{\rho}_d/(1 - \lambda_1 \bar{b}_1), \qquad E(W_1) = \frac{E(N(W_1))}{\lambda_2}, \tag{12}$$

where

$$\bar{\rho}_d = \frac{\lambda_2}{\lambda_1 v^*(\lambda_1) c^*(\lambda_1)} \{(1 + \lambda_1 \bar{s}_3)(1 - c^*(\lambda_1)) + c^*(\lambda_1)[\lambda_1(\bar{v} + \bar{s}_1) + \lambda_1 \bar{b}_1 v^*(\lambda_1)]\}.$$

4 Time Dependent Analysis

Let $N_i(t)$ $i = 1, 2$ be the number of P_i customers in the system at time t and denote by

$$\xi_t = \begin{cases} b_i & \text{if a } P_i \text{ customer in service at } t \quad i = 1,2 \\ s_i & \text{if a } P_i \text{ customer in start up at } t \quad i = 1,2 \\ s_3 & \text{if a } P_1 \text{ customer in special start up at } t \\ c & \text{if the server on close down at } t \\ v & \text{if the server on vacation at } t \\ id & \text{if the server idle at } t \end{cases}$$

and

$$u_t = \begin{cases} 1 & \text{an interrupted } P_2 \text{ customer waits at } t \\ 0 & \text{no interrupted } P_2 \text{ customer waits at } t \end{cases}$$

Let us denote also by $\bar{X}(t)$ the elapsed duration at time t of any random variable X . Define

$$p_{ij}^{(b_k)}(x,t)dx = \Pr[N_1(t) = i, N_2(t) = j, \xi_t = b_k, u_t = 0, x < \bar{B}_k(t) \le x + dx], \quad k = 1, 2$$

$$p_{ij}^{(s_k)}(x,t)dx = \Pr[N_1(t) = i, N_2(t) = j, \xi_t = s_k, u_t = 0, x < \bar{S}_k(t) \le x + dx], \quad k = 1, 2, 3$$

$$p_{ij}^{(c)}(x,t)dx = \Pr[N_1(t) = i, N_2(t) = j, \xi_t = c, u_t = 0, x < \bar{C}(t) \le x + dx],$$

$$p_{ij}^{(v)}(x,t)dx = \Pr[N_1(t) = i, N_2(t) = j, \xi_t = v, u_t = 0, x < \bar{V}(t) \le x + dx],$$

$$q_j^{(id)}(t) = \Pr[N_1(t) = 0, N_2(t) = j, \xi_t = id, u_t = 0],$$

$$P^{(\xi_t)}(s,z_1,z_2,x) = \int_0^\infty e^{-st}\sum_{i=0}^\infty\sum_{j=0}^\infty p_{ij}^{(\xi_t)}(x,t)z_1^i z_2^j dt,$$
$$Q^*(s,z_2) = \int_0^\infty e^{-st}\sum_{j=0}^\infty q_j^{(id)}(t)z_2^j dt,$$

and denote by $\vec{p}_{ij}^{(\xi_t)}(x,t)$, $\vec{P}^{(\xi_t)}(s,z_1,z_2,x)$ the corresponding quantities for $u_t = 1$.Then by connecting as usual the probabilities at t and $t+dt$, forming Laplace Transforms and generating functions and solving the simple differential equations we arrive for $x > 0$ at

$$\begin{aligned}
&P^{(b_k)}(s,z_1,z_2,x) = P^{(b_k)}(s,z_1,z_2,0)(1-B_k(x))\exp[-a(s,z_1,z_2)x], \quad k=1,2\\
&P^{(s_k)}(s,z_1,z_2,x) = P^{(s_k)}(s,z_1,z_2,0)(1-S_k(x))\exp[-a(s,z_1,z_2)x], \quad k=1,2,3\\
&P^{(c)}(s,0,z_2,x) = P^{(c)}(s,0,z_2,0)(1-C(x))\exp[-a(s,0,z_2)x],\\
&P^{(v)}(s,z_1,z_2,x) = P^{(v)}(s,0,z_2,0)(1-V(x))\exp[-a(s,z_1,z_2)x],
\end{aligned} \tag{13}$$

while

$$\begin{aligned}
&\vec{P}^{(b_1)}(s,z_1,z_2,x) = \vec{P}^{(b_1)}(s,z_1,z_2,0)(1-B_1(x))\exp[-a(s,z_1,z_2)x],\\
&\vec{P}^{(s_k)}(s,z_1,z_2,x) = \vec{P}^{(s_k)}(s,z_1,z_2,0)(1-S_k(x))\exp[-a(s,z_1,z_2)x], \quad k=1,3\\
&\vec{P}^{(c)}(s,0,z_2,x) = \vec{P}^{(c)}(s,0,z_2,0)(1-C(x))\exp[-a(s,0,z_2)x],
\end{aligned} \tag{14}$$

and for the idle mode

$$az_2\frac{d}{dz_2}Q^*(s,z_2) + (s+\lambda)Q^*(s,z_2) = 1 + P^{(v)}(s,0,z_2,0)v^*(a(s,0,z_2)). \tag{15}$$

In a similar way we obtain, after algebraic manipulations, for the boundary conditions ($x=0$),

$$\begin{aligned}
[z_1 - b_1^*(a(s,z_1,z_2))]P^{(b_1)}(s,z_1,z_2,0) &= P^{(s_1)}(s,z_1,z_2,0)s_1^*(a(s,z_1,z_2))\\
&+P^{(s_3)}(s,z_1,z_2,0)s_3^*(a(s,z_1,z_2)) - P^{(b_1)}(s,0,z_2,0)b_1^*(a(s,0,z_2))
\end{aligned} \tag{16}$$

$$\begin{aligned}
[z_1 - b_1^*(a(s,z_1,z_2))]\vec{P}^{(b_1)}(s,z_1,z_2,0) &= \vec{P}^{(s_1)}(s,z_1,z_2,0)s_1^*(a(s,z_1,z_2))\\
&+\vec{P}^{(s_3)}(s,z_1,z_2,0)s_3^*(a(s,z_1,z_2)) - \vec{P}^{(b_1)}(s,0,z_2,0)b_1^*(a(s,0,z_2)),
\end{aligned} \tag{17}$$

$$\begin{aligned}
\vec{P}^{(s_1)}(s,z_1,z_2,0) &= \lambda_1 z_1\tfrac{1-s_2^*(a(s,0,z_2))}{a(s,0,z_2)}P^{(s_2)}(s,0,z_2,0),\\
P^{(s_3)}(s,z_1,z_2,0) &= \lambda_1 z_1\tfrac{1-c^*(a(s,0,z_2))}{a(s,0,z_2)}P^{(c)}(s,0,z_2,0),\\
\vec{P}^{(s_3)}(s,z_1,z_2,0) &= \lambda_1 z_1\tfrac{1-c^*(a(s,0,z_2))}{a(s,0,z_2)}\vec{P}^{(c)}(s,0,z_2,0),
\end{aligned} \tag{18}$$

$$
\begin{aligned}
P^{(v)}(s,0,z_2,0) &= P^{(c)}(s,0,z_2,0)c^*(a(s,0,z_2)),\\
\vec{P}^{(c)}(s,0,z_2,0) &= \vec{P}^{(b_1)}(s,0,z_2,0)b_1^*(a(s,0,z_2)),\\
P^{(b_2)}(s,0,z_2,0) &= P^{(s_2)}(s,0,z_2,0)s_2^*(a(s,0,z_2)),
\end{aligned}
\tag{19}
$$

while finally

$$
\begin{aligned}
P^{(c)}(s,0,z_2,0) &= P^{(b_1)}(s,0,z_2,0)b_1^*(a(s,0,z_2) + P^{(b_2)}(s,0,z_2,0)b_2^*(a(s,0,z_2)),\\
P^{(s_1)}(s,z_1,z_2,0) &= \lambda_1 z_1 Q^*(s,z_2) + P^{(v)}(s,0,z_2,0)[v^*(a(s,z_1,z_2))\\
&\quad -v^*(a(s,0,z_2))] + P^{(b_2)}(s,0,z_2,0)[b_2^*(a(s,z_1,z_2)) - b_2^*(a(s,0,z_2))],\\
P^{(s_2)}(s,0,z_2,0) &= a\tfrac{d}{dz_2}Q^*(s,z_2) + \lambda_2 Q^*(s,z_2) + \vec{P}^{(c)}(s,0,z_2,0)c^*(a(s,0,z_2)).
\end{aligned}
\tag{20}
$$

Let us define now

$$
T(s,z_1,z_2) = 1 - K(s,z_1,z_2)c^*(a(s,0,z_2)) - M(s,z_1,z_2),
$$

where the functions K and M have been defined in (8) and (2) respectively. Then by substituting from (18), (20) and (19) to (17) we arrive at

$$
\vec{P}^{(b_1)}(s,z_1,z_2,0) = \frac{K(s,z_1,z_2)\ Q_1^*(s,z_2) - T(s,z_1,z_2)\vec{P}^{(c)}(s,0,z_2,0)}{z_1 - b_1^*(a(s,z_1,z_2))}.
$$

with

$$
Q_1^*(s,z_2) = a\frac{d}{dz_2}Q^*(s,z_2) + \lambda_2 Q^*(s,z_2),
$$

and as the zero of the denominator in $|z_1| \leq 1$, $x(s,z_2)$ say, must be zero of the numerator too, we obtain

$$
\vec{P}^{(c)}(s,0,z_2,0) = \frac{K(s,x(s,z_2),z_2)}{T(s,x(s,z_2),z_2)}Q_1^*(s,z_2)\ , \tag{21}
$$

$$
\vec{P}^{(b_1)}(s,z_1,z_2,0) = \frac{\vec{R}(s,z_1,z_2)}{z_1 - b_1^*(a(s,z_1,z_2))}\quad Q_1^*(s,z_2), \tag{22}
$$

with

$$
\vec{R}(s,z_1,z_2) = K(s,z_1,z_2) - K(s,x(s,z_2),z_2)\frac{T(s,z_1,z_2)}{T(s,x(s,z_2),z_2)}.
$$

Moreover from (18) and (20)

$$
\vec{P}^{(s_3)}(s,z_1,z_2,0) = \frac{M(s,z_1,z_2)K(s,x(s,z_2),z_2)}{T(s,x(s,z_2),z_2)s_3^*(a(s,z_1,z_2))}\ Q_1^*(s,z_2), \tag{23}
$$

$$
P^{(s_2)}(s,0,z_2,0) = R(s,z_2)\ Q_1^*(s,z_2), \tag{24}
$$

with

$$
R(s,z_2) = 1 + c^*(a(s,0,z_2))\frac{K(s,x(s,z_2),z_2)}{T(s,x(s,z_2),z_2)}.
$$

Now from (15)

$$P^{(v)}(s,0,z_2,0) = \frac{Q_2^*(s,z_2)-1}{v^*(a(s,0,z_2))}, \tag{25}$$

with

$$Q_2^*(s,z_2) = az_2\frac{d}{dz_2}Q^*(s,z_2) + (s+\lambda)Q^*(s,z_2)$$

and substituting in (19)

$$P^{(c)}(s,0,z_2,0) = \frac{Q_2^*(s,z_2)-1}{v^*(a(s,0,z_2))c^*(a(s,0,z_2))}. \tag{26}$$

From (18), (19) and (24)

$$\begin{aligned} \vec{P}^{(s_1)}(s,z_1,z_2,0) &= \tfrac{K(s,z_1,z_2)}{s_1^*(a(s,z_1,z_2))}R(s,z_2)\ Q_1^*(s,z_2), \\ P^{(b_2)}(s,0,z_2,0) &= s_2^*(a(s,0,z_2))\ R(s,z_2)\ Q_1^*(s,z_2), \\ P^{(s_3)}(s,z_1,z_2,0) &= \tfrac{M(s,z_1,z_2)}{s_3^*(a(s,z_1,z_2))}\ \tfrac{Q_2^*(s,z_2)-1}{v^*(a(s,0,z_2))c^*(a(s,0,z_2))}. \end{aligned} \tag{27}$$

Substituting finally from (20) and (23) in (16) and denoting

$$\begin{aligned} h_1(s,z_1,z_2) &= aL(s,z_1,z_2)\ R(s,z_2) - \ az_2/e(s,z_1,z_2), \\ h_2(s,z_1,z_2) &= \lambda_1 z_1 s_1^*(a(s,z_1,z_2)) + \lambda_2 L(s,z_1,z_2)\ R(s,z_2) - \ (s+\lambda)/e(s,z_1,z_2), \end{aligned} \tag{28}$$

we arrive at

$$P^{(b_1)}(s,z_1,z_2,0) = \frac{h_1(s,z_1,z_2)\frac{d}{dz_2}Q^*(s,z_2) + \ h_2(s,z_1,z_2)Q^*(s,z_2)] + 1/\ e(s,z_1,z_2)}{z_1 - b_1^*(a(s,z_1,z_2))}, \tag{29}$$

and using the zero of the denominator in the unit disk we obtain

$$a(z_2 - D(s,z_2))\frac{d}{dz_2}Q^*(s,z_2) + F(s,z_2)Q^*(s,z_2) = 1, \tag{30}$$

where now

$$\begin{aligned} D(s,z_2) &= \ L(s,x(s,z_2),z_2)\ R(s,z_2)e(s,x(s,z_2),z_2), \\ F(s,z_2) &= s + \lambda - \ \lambda_1 x(s,z_2)s_1^*(a(s,x(s,z_2),z_2))e(s,x(s,z_2),z_2) - \lambda_2 D(s,z_2) \\ &= s + \lambda_1(1 - d^*(s,z_2)) + \lambda_2(1 - D(s,z_2)). \end{aligned} \tag{31}$$

We have to state here the following theorem

Theorem 1 *For (i)* $Re(s) > 0$, $|w| \leq 1$ *(ii)* $Re(s) \geq 0, |w| < 1$ *and (iii)* $Re(s) \geq 0, |w| \leq 1$ *and*

$$\rho = \lambda_1\bar{b}_1 + \bar{\rho}_v + \bar{\rho}_c > 1 \tag{32}$$

the equation

$$z_2 - wD(s,z_2) = 0 \tag{33}$$

has one and only one root, $z_2 = \phi(s, w)$ say, inside the region $|z_2| < 1$. Specifically for $s = 0$ and $w = 1$, $\phi(0, 1)$ is the smallest positive real root of (33) with $\phi(0, 1) < 1$ if $\rho > 1$ and $\phi(0, 1) = 1$ for $\rho \leq 1$.

Proof: Comparing $D(s, z_2)$ in the first of (31) with the generating function $w^*(s, z_2)$ in (9) of section 3 one realizes easily that

$$D(s, z_2) = w^*(s, z_2) = \int_0^\infty e^{-st} \sum_{j=0}^\infty w_j(t) z_2^j dt,$$

i.e. $D(s, z_2)$ is in fact the Laplace transform of a generating function.

Thus for the closed contour $|z_2| = 1 - \epsilon$ ($\epsilon > 0$ is a small number) and under the assumptions (i) and (ii) we can always find a sufficiently small $\epsilon \geq 0$ such that

$$|wD(s, z_2)| \leq |w| D(Re(s), 1 - \epsilon) < 1 - \epsilon, \tag{34}$$

while for $Re(s) \geq 0$, $|w| \leq 1$ we need in addition

$$\frac{d}{d\epsilon} D(0, 1 - \epsilon) \mid_{\epsilon=0} < -1,$$

or $\rho > 1$ for the relation (34) to hold. A final reference to Rouche's theorem completes the first part of the proof.

Moreover for $s = 0$ and $w = 1$ the convex function $D(0, z_2)$ is a monotonically increasing function of z_2, for $0 \leq z_2 \leq 1$, taking the values $0 < D(0, 0) < 1$ and $D(0, 1) = 1$ and so $0 < \phi(0, 1) < 1$ if $\rho > 1$, while for $\rho \leq 1$, $\phi(0, 1)$ becomes equal to 1 and this completes the proof. □

Using the theorem above one can solve (see Falin & Fricker [7]) the differential equation (30) and obtain

$$Q^*(s, z_2) = \frac{1}{F(s, z_2)}, \qquad if\ z_2 = \phi(s, 1),$$

$$Q^*(s, z_2) = \int_{z_2}^{\phi(s,1)} \frac{1}{a(D(s, u) - u)} \exp\{\int_u^{z_2} \frac{F(s, x)}{a(D(s, x) - x)} dx\} du, \quad if\ z_2 \neq \phi(s, 1).$$

Thus the quantity $Q^*(s, z_2)$ is known and so from the second of (20) and (21)- (29) all generating functions are completely known. This completes the time-dependent analysis of the model.

5 Stability Conditions

For a stochastic process $(Y(t)\ ; t \geq 0)$ we will say that it is stable, if its limiting probabilities as $t \to \infty$ exist and form a distribution.

Consider now the points T_n in time at which, either a generalized busy period of P_1 customers, or a generalized completion time of a P_2 customer is finished, i.e. the points at which the server becomes idle. If

$$0 = T_0 < T_1 < T_2 < \cdots,$$

is the sequence of these points in ascending order and define $\zeta_n = N_2(T_n + 0)$, then it is easy to understand that the stochastic process $Z = (\zeta_n; n \geq 0)$ is an irreducible and aperiodic Markov chain. Then

Theorem 2 *For $\rho < 1$ the Markov chain Z is positive recurrent.*

Proof. To prove the theorem, we will use the following criterion (see Pakes [8]):

An irreducible and aperiodic Markov chain $(Y_n \; ; \; n \geq 0)$, with state space the nonnegative integers, is positive recurrent if $|\delta_k| < \infty$ for all $k = 0, 1, 2, \ldots$ and $\limsup_{k \to \infty} \delta_k < 0$, where $\delta_k = E[Y_{n+1} - Y_n \mid Y_n = k]$.

For the Markov chain Z of our model, let

$$h_{k,m}(t)dt = \Pr[t < T_{n+1} - T_n \leq t + dt,\; N_2(T_{n+1}) - N_2(T_n) = m |\; N_2(T_n) = k].$$

Then it is easy to see that for $m = 0, 1, 2, \ldots$

$$\begin{aligned} h_{k,m}(t) = \;& \lambda_1 e^{-(\lambda_1+\lambda_2+ka)t} * d_m(t) + \lambda_2 e^{-(\lambda_1+\lambda_2+ka)t} * w_m(t) \\ & + ka e^{-(\lambda_1+\lambda_2+ka)t} * w_{m+1}(t), \end{aligned}$$

while for $m = -1$

$$h_{k,-1}(t) = \; ka e^{-(\lambda_1+\lambda_2+ka)t} * w_0(t),$$

and so

$$\int_0^\infty e^{-st} \sum_{m=-1}^{\infty} h_{k,m}(t) z^m dt = \frac{\lambda_1 d^*(s,z) + \lambda_2 w^*(s,z) + \frac{ka}{z} w^*(s,z)}{s + \lambda_1 + \lambda_2 + ka}, \qquad (35)$$

and by taking derivatives above with respect to z at the point $(z = 1, s = 0)$ we arrive at

$$\delta_k = \frac{\lambda_1 E(N(W_1)) + \lambda_2 E(N(W_2)) + ka[E(N(W_2)) - 1]}{\lambda_1 + \lambda_2 + ka}, \qquad k = 0, 1, \ldots .$$

where $E(N(W_1))$, $E(N(W_2))$ have been found in (10) and (12) respectively.

Thus for $\rho < 1$ we realize that $|\delta_k|$ is finite for all k and also $\limsup_{k \to \infty} \delta_k = E(N(W_2)) - 1 = \frac{\rho - 1}{1 - \lambda_1 b_1} < 0$, and the criterion is satisfied. □

Consider now the stochastic process

$$\mathbf{Z} = \{(N_1(t),\; N_2(t),\; \xi_t) : \; 0 \leq t < \infty\}$$

where $N_i(t)$, ξ_t have been defined in section 4. Then

Theorem 3 *For $\rho < 1$ the process $\mathbf{Z}$ is stable.*

Proof: Consider the quantity

$$m_k = E(T_1|\ \zeta_0 = k)$$

By taking derivatives in (35) with respect to s (at $z = 1$) we obtain

$$m_k = \frac{\lambda_1 E(W_1) + \lambda_2 E(W_2) + kaE(W_2) + 1}{\lambda_1 + \lambda_2 + ka},$$

and if $q_k \quad k = 0, 1, 2, ...$ are the steady state probabilities of the positive recurrent Markov chain Z then

$$\mathbf{q}\cdot\mathbf{m} = \sum_{k=0}^{\infty} q_k m_k = E(W_2) + \{1 + \lambda_1[E(W_1) - E(W_2)]\} \sum_{k=0}^{\infty} \frac{q_k}{\lambda_1 + \lambda_2 + ka}. \quad (36)$$

Now it is clear that there is always a finite integer k^* such that

$$\frac{1}{\lambda_1 + \lambda_2 + (k^* - 1)a} > 1 > \frac{1}{\lambda_1 + \lambda_2 + k^* a},$$

and so

$$\sum_{k=0}^{\infty} \frac{q_k}{\lambda_1+\lambda_2+ka} = \sum_{k=0}^{k^*-1} \frac{q_k}{\lambda_1+\lambda_2+ka} + \sum_{k=k^*}^{\infty} \frac{q_k}{\lambda_1+\lambda_2+ka} < \sum_{k=0}^{k^*-1} \frac{q_k}{\lambda_1+\lambda_2+ka}$$
$$+ \sum_{k=k^*}^{\infty} q_k = \sum_{k=0}^{k^*-1} \frac{q_k}{\lambda_1+\lambda_2+ka} + (1 - \sum_{k=0}^{k^*-1} q_k) < \infty$$

and so from (36) using (10), (12) we understand that $\mathbf{q}\cdot\mathbf{m} < \infty$.

Consider finally the irreducible aperiodic and positive recurrent Markov Renewal Process $\{Z,\ T\} = \{(\zeta_n,\ T_n) :\ n = 0, 1, 2, ...\}$. It is easy to see that the stochastic process $\mathbf{Z}$ is a Semi-Regenerative Process with imbedded Markov Renewal Process $\{Z,\ T\}$ and as $\mathbf{q}\cdot\mathbf{m} < \infty$ it is clear that $\mathbf{Z}$ is stable (Cinlar [9], Theorem 6.12 p.347). □

6 Steady State Probabilities

Suppose now that $\rho < 1$. Let

$$p_{ij}^{(\xi_t)}(x) = \lim_{t\to\infty} p_{ij}^{(\xi_t)}(x,t), \qquad q_j^{(id)} = \lim_{t\to\infty} q_j^{(id)}(t), \qquad Q^*(z_2) = \sum_{j=0}^{\infty} q_j^{(id)} z_2^j,$$

$$P^{(\xi_t)}(z_1, z_2, x) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} p_{ij}^{(\xi_t)}(x) z_1^i z_2^j, \qquad P^{(\xi_t)}(z_1, z_2) = \int_0^{\infty} P^{(\xi_t)}(z_1, z_2, x)dx,$$

and denote by $\vec{p}_{ij}^{(\xi_t)}(x)$, $\vec{P}^{(\xi_t)}(z_1, z_2, x)$, $P^{(\xi_t)}(z_1, z_2)$ the corresponding quantities for $u_t = 1$. Then it is well known that

$$p_{ij}^{(\xi_t)}(x) = \lim_{t\to\infty} p_{ij}^{(\xi_t)}(x,t) = \lim_{s\to 0} s \int_0^{\infty} e^{-st} p_{ij}^{(\xi_t)}(x,t)dt,$$

$$q_j^{(id)} = \lim_{t\to\infty} q_j^{(id)}(t) = \lim_{s\to 0} s \int_0^\infty e^{-st} q_j^{(id)}(t)dt,$$

and so integrating with respect to x, multiplying by s and taking limits $s \to \infty$ in (13), (14) we arrive at

$$\begin{aligned}
&P^{(b_k)}(z_1,z_2) = P^{(b_k)}(z_1,z_2,0)[1-b_k^*(a(0,z_1,z_2))]/a(0,z_1,z_2), \qquad k=1,2\\
&P^{(s_k)}(z_1,z_2) = P^{(s_k)}(z_1,z_2,0)\ [1-s_k^*(a(0,z_1,z_2))]/a(0,z_1,z_2), \qquad k=1,2,3\\
&P^{(c)}(0,z_2) = P^{(c)}(0,z_2,0)\ [1-c^*(a(0,z_1,z_2))]/a(0,0,z_2),\\
&P^{(v)}(z_1,z_2) = P^{(v)}(0,z_2,0)\ [1-v^*(a(0,z_1,z_2))]/a(0,z_1,z_2),
\end{aligned} \tag{37}$$

and

$$\begin{aligned}
&\vec{P}^{(b_1)}(z_1,z_2) = \vec{P}^{(b_1)}(z_1,z_2,0)\ [1-b_1^*(a(0,z_1,z_2))]/a(0,z_1,z_2),\\
&\vec{P}^{(s_k)}(z_1,z_2) = \vec{P}^{(s_k)}(z_1,z_2,0)\ [1-s_k^*(a(0,z_1,z_2))]/a(0,z_1,z_2), \qquad k=1,3\\
&\vec{P}^{(c)}(0,z_2) = \vec{P}^{(c)}(0,z_2,0)\ [1-c^*(a(0,z_1,z_2))]/a(0,0,z_2).
\end{aligned} \tag{38}$$

In a similar way we obtain for the boundary conditions

$$\begin{aligned}
\vec{P}^{(c)}(0,z_2,0) &= \tfrac{K(0,x(0,z_2),z_2)}{T(0,x(0,z_2),z_2)}\ Q_1^*(z_2),\\
\vec{P}^{(b_1)}(z_1,z_2,0) &= \tfrac{\vec{R}(0,z_1,z_2)}{z_1-b_1^*(a(0,z_1,z_2))}\ Q_1^*(z_2),\\
\vec{P}^{(s_3)}(z_1,z_2,0) &= \tfrac{M(0,z_1,z_2)K(0,x(0,z_2),z_2)}{T(0,x(0,z_2),z_2)s_3^*(a(0,z_1,z_2))}\ Q_1^*(z_2),\\
P^{(s_2)}(0,z_2,0) &= R(0,z_2)\ Q_1^*(z_2),\\
\vec{P}^{(s_1)}(z_1,z_2,0) &= \tfrac{K(0,z_1,z_2)}{s_1^*(a(0,z_1,z_2))} R(0,z_2)\ Q_1^*(z_2),\\
P^{(b_2)}(0,z_2,0) &= s_2^*(a(0,0,z_2))\ R(0,z_2)\ Q_1^*(z_2),
\end{aligned} \tag{39}$$

$$\begin{aligned}
&P^{(v)}(0,z_2,0) = \tfrac{Q_2^*(z_2)}{v^*(a(0,0,z_2))},\\
&P^{(c)}(0,z_2,0) = \tfrac{Q_2^*(z_2)}{v^*(a(0,0,z_2))c^*(a(0,0,z_2))},\\
&P^{(s_3)}(z_1,z_2,0) = \tfrac{M(0,z_1,z_2)}{s_3^*(a(0,z_1,z_2))}\ \tfrac{Q_2^*(z_2)}{v^*(a(0,0,z_2))c^*(a(0,0,z_2))},
\end{aligned} \tag{40}$$

with

$$Q_1^*(z_2) = a\frac{d}{dz_2}Q^*(z_2) + \lambda_2 Q^*(z_2), \qquad Q_2^*(z_2) = az_2\frac{d}{dz_2}Q^*(z_2) + \lambda Q^*(z_2)$$

$$\begin{aligned}
P^{(s_1)}(z_1,z_2,0) = {} & \lambda_1 z_1 Q^*(z_2) + P^{(v)}(0,z_2,0)[v^*(a(0,z_1,z_2)) - v^*(a(0,0,z_2))]\\
& + P^{(b_2)}(0,z_2,0)[b_2^*(a(0,z_1,z_2)) - b_2^*(a(0,0,z_2))],
\end{aligned} \tag{41}$$

$$P^{(b_1)}(z_1, z_2, 0) = \frac{h_1(0, z_1, z_2)\frac{d}{dz_2}Q^*(z_2) + \ h_2(0, z_1, z_2)Q^*(z_2)]}{z_1 - b_1^*(a(0, z_1, z_2))}, \tag{42}$$

while the differential equation (30) becomes

$$a(z_2 - D(0, z_2))\frac{d}{dz_2}Q^*(z_2) + F(0, z_2)Q^*(z_2) = 0, \tag{43}$$

with $D(0, z_2) = w^*(0, z_2)$. From (31)

$$F(0, z_2) = \lambda_1(1 - d^*(0, z_2)) + \lambda_2(1 - D(0, z_2)) = \lambda[1 - G(z_2)],$$

where

$$G(z_2) = \frac{\lambda_1 d^*(0, z_2) + \lambda_2 w^*(0, z_2)}{\lambda}.$$

Let now

$$\omega(z_2) = \frac{1 - G(z_2)}{z_2 - w^*(0, z_2)},$$

then for $\rho < 1$ the quantity $z_2 - w^*(0, z_2)$ never becomes zero in $|z_2| < 1$ (Theorem 1) and also

$$\lim_{z_2 \to 1} \omega(z_2) = -\frac{\frac{\lambda_1}{\lambda}\bar{\rho}_d + \frac{\lambda_2}{\lambda}(\bar{\rho}_v + \bar{\rho}_c)}{1 - \rho} < \infty.$$

Thus $\omega(z_2)$ is an analytic function in $|z_2| < 1$ and a continuous one on the boundary and so for any $|z_2| \leq 1$ we can solve equation (43) and obtain

$$Q^*(z_2) = Q^*(1)\exp\{-\frac{\lambda}{a}\int_{z_2}^{1} \frac{1 - G(u)}{w^*(0, u) - u}du\},$$

Replacing finally $Q^*(z_2)$ back in the generating functions and asking for the total probabilities to sum to unity we arrive at

$$Q^*(1) = \frac{1 - \rho}{1 - \lambda_1\bar{b}_1 + \frac{\lambda_1}{\lambda_2}\bar{\rho}_d}.$$

and so the generating functions of the steady state probabilities are completely known.

The following theorem shows that the condition $\rho < 1$ is also necessary for a stable system.

Theorem 4 *If the stochastic process* $\mathbf{Z}$ *is stable then* $\rho < 1$.

Proof: Suppose that $\mathbf{Z}$ stable and $\rho > 1$. Then from theorem 1 the equation $z_2 - D(0, z_2) = 0$ has a root $0 < \phi(0, 1) < 1$ and

$$F(0, \phi(0, 1)) = \lambda_1(1 - d^*(0, \phi(0, 1))) + \lambda_2(1 - D(0, \phi(0, 1))) \neq 0.$$

By putting now $\phi(0,1)$ instead of z_2 in (43) we obtain

$$F(0,\phi(0,1))Q^*(\phi(0,1)) = 0,$$

and so $Q^*(\phi(0,1)) = \sum q_j^{(id)}\phi^j(0,1) = 0$ with $0 < \phi(0,1) < 1$. Thus $q_j^{(id)} = 0\ \forall\, j$ and also from the generating functions in (37)- (42) it is clear that all probabilities become zero. This of course contradicts to the hypothesis that the system is stable.

Suppose finally that $\mathbf{Z}$ stable and $\rho = 1$. By taking derivatives with respect to z_2 in (43) (at $z_2 = 1$) we arrive (for $\rho = 1$) at

$$\frac{d}{dz_2}F(0,z_2)|_{z_2=1}Q^*(1) = -[\lambda_1 E(N(W_1)) + \lambda_2 E(N(W_2))]Q^*(1) = 0,$$

and so $Q^*(1) = \sum q_j^{id} = 0$ and this again contradicts to the hypothesis that the system is stable. □

7 Performance Measures

7.1 Probabilities of server state

In this section we will use formulas for the generating functions obtained previously, to derive expressions for the probabilities of server state. Thus by putting $z_1 = z_2 = 1$ into relations (37)-(42) we obtain easily

$P[server\ Idle] =$	$P(\ \xi = id) = Q^*(1) = \frac{1-\rho}{1-\lambda_1\bar{b}_1+\frac{\lambda_1}{\lambda_2}\bar{\rho}_d}$
$P[a\ P_1\ customer\ in\ service] =$	$P(\ \xi = b_1) = \lambda_1\bar{b}_1$
$P[a\ P_2\ customer\ in\ service] =$	$P(\ \xi = b_2) = \lambda_2\bar{b}_2$
$P[server\ in\ P_2\ start\ up] =$	$P(\ \xi = s_2) = \frac{\lambda_2}{s_2^*(\lambda_1)}\frac{1-s_2^*(\lambda_1)}{\lambda_1}$
$P[server\ in\ vacation] =$	$P(\xi = v) = \frac{\lambda_2+\lambda_1 Q^*(1)}{v^*(\lambda_1)}\bar{v}$
$P[server\ in\ close\ down] =$	$P(\ \xi = c) = \frac{1-c^*(\lambda_1)}{\lambda_1 c^*(\lambda_1)}[\lambda_2\frac{1-s_2^*(\lambda_1)}{s_2^*(\lambda_1)} + \frac{\lambda_2+\lambda_1 Q^*(1)}{v^*(\lambda_1)}]$
$P[server\ in\ special\ start\ up] =$	$P(\ \xi = s_3) = \lambda_1\bar{s}_3 P(\xi\ = c)$
$P[server\ in\ P_1\ start\ up] =$	$P(\xi = s_1) = \bar{s}_1[\frac{\lambda_2+\lambda_1 Q^*(1)}{v^*(\lambda_1)} - \lambda_2 b_2^*(\lambda_1) + \lambda_2\frac{1-s_2^*(\lambda_1)}{s_2^*(\lambda_1)}]$

7.2 Mean number of ordinary customers

For any p.d.f. $a(t)$, let us denote now $\bar{a}^{(2)} = \int_0^\infty t^2 a(t)dt$, i.e., denote by $\bar{a}^{(2)}$ its second moment about zero. By differentiating the generating functions with respect to z_1 at the point $z_1 = z_2 = 1$ we obtain the mean number of P_1 customers, according to the server state, as following,

$$E(N_1;\ \xi = v) = \frac{\lambda_1\bar{v}^{(2)}}{2v^*(\lambda_1)}(\lambda_2 + \lambda_1 Q^*(1))$$

$$E(N_1;\xi=s_3)=(1-c^*(\lambda_1))\,(\bar{s}_3+\frac{\lambda_1\bar{s}_3^{(2)}}{2})(\frac{\lambda_2+\lambda_1Q^*(1)}{c^*(\lambda_1)\,v^*(\lambda_1)}+\lambda_2\frac{1-s_2^*(\lambda_1)}{c^*(\lambda_1)\,s_2^*(\lambda_1)})$$

$$\begin{aligned}E(N_1;\xi=s_1) &= \lambda_1\bar{s}_1[Q^*(1)+\frac{\bar{v}}{v^*(\lambda_1)}(\lambda_2+\lambda_1Q^*(1))+\lambda_2\bar{b}_2]+\frac{\lambda_1\bar{s}_1^{(2)}}{2}\\ &\times[\frac{\lambda_2+\lambda_1Q^*(1)}{v^*(\lambda_1)}-\lambda_2b_2^*(\lambda_1)]+\frac{\lambda_2(1-s_2^*(\lambda_1))\,(\bar{s}_1+\lambda_1\frac{\bar{s}_1^{(2)}}{2})}{s_2^*(\lambda_1)}\end{aligned}$$

$$E(N_1;\xi=b_2)=\frac{\lambda_1\lambda_2}{2}\bar{b}_2^{(2)}$$

$$E(N_1;\xi=b_1)=\frac{\lambda_1\bar{b}_1}{2(1-\lambda_1\bar{b}_1)}\left(\lambda_1\bar{b}_1^{(2)}/\bar{b}_1+A_1+A_2\right)$$

where

$$A_1=\frac{\lambda_2(1-s_2^*(\lambda_1))}{s_2^*(\lambda_1)\,c^*(\lambda_1)}\left[(1-c^*(\lambda_1))\left(\lambda_1\bar{s}_3^{(2)}+2\bar{s}_3\right)+c^*(\lambda_1)\left(\lambda_1\bar{s}_1^{(2)}+2\bar{s}_1\right)\right]$$

$$\begin{aligned}A_2 &= \lambda_1(\lambda_1\bar{s}_1^{(2)}+2\bar{s}_1)Q^*(1)+\frac{\lambda_2+\lambda_1Q^*(1)}{c^*(\lambda_1)\,v^*(\lambda_1)}\Big[(1-c^*(\lambda_1))\,(\lambda_1\bar{s}_3^{(2)}+2\bar{s}_3)\\ &+\lambda_1c^*(\lambda_1)\,(\bar{s}_1^{(2)}(1-v^*(\lambda_1))+2\bar{s}_1\bar{v}+\bar{v}^{(2)})\Big]\\ &+\lambda_1\lambda_2(\bar{s}_1^{(2)}(1-b_2^*(\lambda_1))+2\bar{s}_1\bar{b}_2+\bar{b}_2^{(2)})\end{aligned}$$

7.3 Mean number of retrial customers

To derive expressions for the mean number of customers in the retrial box we need firstly to calculate the derivatives of some functions defined in previous sections. Let

$$U(0,x(0,z_2),z_2)=\frac{K(0,x(0,z_2),z_2)}{T(0,x(0,z_2),z_2)},\quad U=U(0,1,1)=\frac{1-s_2^*(\lambda_1)}{c^*(\lambda_1)s_2^*(\lambda_1)},$$

$$Q^{*(1)}=\tfrac{d}{dz_2}\,Q^*(z_2)|_{z_2=1}=\tfrac{\lambda_2}{a}(\tfrac{\lambda_1\bar{\rho}_d+\lambda_2(\bar{\rho}_c+\bar{\rho}_v)}{\lambda_1\bar{\rho}_d+\lambda_2(1-\lambda_1\bar{b}_1)}),$$

$$Q^{*(2)}=\tfrac{d^2}{dz_2^2}\,Q^*(z_2)|_{z_2=1}=\tfrac{1-\lambda_1\bar{b}_1}{2a(1-\rho)}[\lambda_2\rho_w^{(2)}+\lambda_1\rho_d^{(2)}Q^*(1)],$$

$$H_1=aQ^{*(2)}+\lambda_2Q^{*(1)},\quad H_2=aQ^{*(2)}+(\lambda+a)\,Q^{*(1)}$$

where $\rho_d^{(2)}$ and $\rho_w^{(2)}$ are given below in (44) and (45). For any function $f^*(z_2)$ denote $f^{*(\nu)}=\frac{d^\nu}{dz_2^\nu}\,f^*(z_2)|_{z_2=1}$ and

$$\sigma_{f^*}(z_2)=\left(\tfrac{1-f^*(a(0,0,z_2))}{a(0,0,z_2)}\right),\qquad \sigma_{f^*}=\sigma_{f^*}(1)=\tfrac{1-f^*(\lambda_1)}{\lambda_1},$$

$$\sigma_{f^*}^{(1)}=\tfrac{d}{dz_2}\,\sigma_{f^*}(z_2)|_{z_2=1}=\tfrac{\lambda_1\lambda_2f^{*(1)}(\lambda_1)+\lambda_2(1-f^*(\lambda_1))}{\lambda_1^2},$$

$$\sigma_f^{(2)}=\tfrac{d^2}{dz_2^2}\,\sigma_{f^*}(z_2)|_{z_2=1}=\tfrac{-\lambda_1^2\lambda_2^2f^{*(2)}(\lambda_1)+2\lambda_2\left(\lambda_1\lambda_2f^{*(1)}(\lambda_1)+\lambda_2(1-f^*(\lambda_1))\right)}{\lambda_1^3}$$

while for any function $w(0, x(0, z_2), z_2)$ denote $\hat{w}^{(v)} = \frac{d^{\nu}}{dz_2^{\nu}} w(0, x(0, z_2), z_2)|_{z_2=1}$ .Then

$$\hat{K}^{(1)} = \lambda_2 s_2^{*(1)}(\lambda_1) + \frac{\lambda_2}{1-\lambda_1 \bar{b}_1} \sigma_{s_2^*}(1 + \lambda_1 \bar{s}_1),$$

$$\hat{M}^{(1)} = \lambda_2 c^{*(1)}(\lambda_1) + \frac{\lambda_2}{1-\lambda_1 \bar{b}_1} \sigma_{c^*}(1 + \lambda_1 \bar{s}_3),$$

$$\hat{T}^{(1)} = -\hat{K}^{(1)} c^*(\lambda_1) + \lambda_1 \lambda_2 c^{*(1)}(\lambda_1)\ \sigma_{s_2^*} - \hat{M}^{(1)},$$

$$\hat{U}^{(1)} = \{\hat{K}^{(1)} c^*(\lambda_1) s_2^*(\lambda_1) - \hat{T}^{(1)} \lambda_1\ \sigma_{s_2^*}\} / (c^*(\lambda_1) s_2^*(\lambda_1))^2,$$

$$\hat{e}^{(1)} = \frac{\lambda_2}{\lambda_1(1-\lambda_1 \bar{b}_1)} \left[\frac{\lambda_1\ \sigma_{c^*}(1+\lambda_1 \bar{s}_3) + \lambda_1 c^*(\lambda_1)(\bar{v} + \bar{s}_1 \lambda_1\ \sigma_{v^*})}{c^*(\lambda_1) v^*(\lambda_1)}\right],$$

$$\hat{L}^{(1)} = -\lambda_2 s_2^{*(1)}(\lambda_1) + \frac{\lambda_2}{1-\lambda_1 \bar{b}_1} s_2^*(\lambda_1)\left(\bar{b}_2 + \bar{s}_1 \lambda_1\ \sigma_{b_2^*}\right),$$

$$R^{(1)} = \frac{d}{dz_2} R(0, z_2)|_{z_2=1} = c^*(\lambda_1) \hat{U}^{(1)} - \frac{\lambda_2 c^{*(1)}(\lambda_1) \lambda_1\ \sigma_{s_2^*}}{c^*(\lambda_1) s_2^*(\lambda_1)}.$$

where the functions K, M, T, e, L, R, have been defined in sections 3 and 4. Using the above quantities we have for the retrial customers the following results.

$$E(N_2; \xi = v) = \frac{\bar{v}}{v^*(\lambda_1)} H_2 + \frac{\lambda_2 + \lambda_1 Q^*(1)}{v^*(\lambda_1)} \lambda_2 [\frac{\bar{v}^{(2)}}{2} + \frac{v^{*(1)}(\lambda_1)}{v^*(\lambda_1)} \bar{v}],$$

$$E(N_2; \xi = s_2) = \sigma_{s_2^*}[\lambda_2 R^{(1)} + \frac{H_1}{s_2^*(\lambda_1)}] + \frac{\lambda_2}{s_2^*(\lambda_1)} \sigma_{s_2^*}^{(1)},$$

$$E(N_2; \xi = b_2) = \bar{b}_2 [s_2^*(\lambda_1)(\lambda_2 R^{(1)} + \frac{H_1}{s_2^*(\lambda_1)}) - \lambda_2^2 \frac{s_2^{*(1)}(\lambda_1)}{s_2^*(\lambda_1)}] + \frac{\lambda_2^2}{2} \bar{b}_2^{(2)},$$

$$E(N_2; \xi \quad = c) = \sigma_{c^*}\{[\lambda_2 U^{(1)} + H_1 U] + \frac{1}{c^*(\lambda_1) v^*(\lambda_1)} [H_2 + \frac{(\lambda_2 + \lambda_1 Q^*(1))}{c^*(\lambda_1) v^*(\lambda_1)} \lambda_2$$
$$\times \left(c^{*(1)}(\lambda_1) v^*(\lambda_1) + c^*(\lambda_1) v^{*(1)}(\lambda_1)\right)]\} + \sigma_{c^*}^{(1)} [\lambda_2 U + \frac{(\lambda_2 + \lambda_1 Q^*(1))}{c^*(\lambda_1) v^*(\lambda_1)}],$$

$$E(N_2; \xi = s_3) = \lambda_1 \{\bar{s}_3 E(N_2; \xi = c) + \frac{\lambda_2}{2} \bar{s}_3^{(2)} P(\xi = c)\}$$

$$E(N_2; \xi = s_1, u = 1) = \quad \lambda_1 \bar{s}_1 [\frac{\lambda_2}{s_2^*(\lambda_1)} \sigma_{s_2^*}^{(1)} + \sigma_{s_2^*}(\lambda_2 R^{(1)} + \frac{H_1}{s_2^*(\lambda_1)})] + \frac{\lambda_2^2 \lambda_1}{2 s_2^*(\lambda_1)} \bar{s}_1^{(2)} \sigma_{s_2^*},$$

$$E(N_2; \xi = s_1 \quad, u = 0) = \frac{\lambda_2 \bar{s}_1^{(2)}}{2} [\frac{\lambda_2 + \lambda_1 Q^*(1)}{v^*(\lambda_1)} - \lambda_2 b_2^*(\lambda_1)] + \bar{s}_1 \{\ \frac{\lambda_2 + \lambda_1 Q^*(1)}{v^*(\lambda_1)} \lambda_2 [\bar{v}$$
$$+ v^{*(1)}(\lambda_1)] + \lambda_1 Q^{*(1)} + \frac{\lambda_1 \sigma_{v^*}}{v^*(\lambda_1)} [H_2 + \frac{\lambda_2 v^{*(1)}(\lambda_1)}{v^*(\lambda_1)} (\lambda_2 + \lambda_1 Q^*(1))]$$
$$+ \lambda_1 \sigma_{b_2^*} [s_2^*(\lambda_1)(\lambda_2 R^{(1)} + \frac{H_1}{s_2^*(\lambda_1)}) - \frac{\lambda_2^2 s_2^{*(1)}(\lambda_1)}{s_2^*(\lambda_1)}] + \lambda_2^2 (\bar{b}_2 + b_2^{*(1)}(\lambda_1))\}.$$

Except for the quantities introduced before we need also the second derivatives to proceed to $E(N_2;\xi=b_1)$. Thus

$$\begin{aligned}
\hat{K}^{(2)} = {} & 2\frac{\lambda_2^2 s_2^{*(1)}(\lambda_1)}{\lambda_1\left(1-\lambda_1\bar{b}_1\right)}\left(1+\lambda_1\bar{s}_1\right)+\frac{2\lambda_2^2}{\lambda_1\left(1-\lambda_1\bar{b}_1\right)}\sigma_{s_2^*}-\lambda_2^2 s_2^{*(2)}\left(\lambda_1\right) \\
& +\frac{\lambda_2^2\sigma_{s_2^*}}{(1-\lambda_1\bar{b}_1)^2}[(\lambda_1\bar{s}_1^{(2)}+2\bar{s}_1)+\frac{\lambda_1\bar{b}_1^{(2)}}{1-\lambda_1\bar{b}_1}\left(1+\lambda_1\bar{s}_1\right)], \\
\hat{M}^{(2)} = {} & 2\frac{\lambda_2^2 c^{*(1)}(\lambda_1)}{\lambda_1\left(1-\lambda_1\bar{b}_1\right)}\left(1+\lambda_1\bar{s}_3\right)+2\frac{\lambda_2^2}{\lambda_1\left(1-\lambda_1\bar{b}_1\right)}\sigma_{c^*}-\lambda_2^2 c^{*(2)}\left(\lambda_1\right) \\
& +\frac{\lambda_2^2\sigma_{c^*}}{(1-\lambda_1\bar{b}_1)^2}[(\lambda_1\bar{s}_3^{(2)}+2\bar{s}_3)+\frac{\lambda_1\bar{b}_1^{(2)}}{1-\lambda_1\bar{b}_1}\left(1+\lambda_1\bar{s}_3\right)], \\
\hat{L}^{(2)} = {} & \lambda_2^2 s_2^{*(2)}\left(\lambda_1\right)-2\frac{\lambda_2^2 s_2^{*(1)}(\lambda_1)(\bar{b}_2+\bar{s}_1)}{1-\lambda_1\bar{b}_1}+2\frac{\lambda_2^2\bar{s}_1(s_2^{*(1)}(\lambda_1)b_2^*(\lambda_1)+s_2^*(\lambda_1)b_2^{*(1)}(\lambda_1))}{1-\lambda_1\bar{b}_1} \\
& +\frac{\lambda_2^2 s_2^*(\lambda_1)}{(1-\lambda_1\bar{b}_1)^2}[\frac{\lambda_1\bar{b}_1^{(2)}}{1-\lambda_1\bar{b}_1}\left(\bar{b}_2+\lambda_1\bar{s}_1\sigma_{b_2^*}\right)+(\bar{b}_2^{(2)}+\lambda_1\bar{s}_1^{(2)}\sigma_{b_2^*}+2\bar{s}_1\bar{b}_2)], \\
\hat{e}^{(2)} = {} & \hat{J}+2\left(\frac{\lambda_2\left(c^{*(1)}(\lambda_1)v^*(\lambda_1)+c^*(\lambda_1)v^{*(1)}(\lambda_1)\right)}{c^*(\lambda_1)v^*(\lambda_1)}+\hat{e}^{(1)}\right)\hat{e}^{(1)}
\end{aligned}$$

where

$$\begin{aligned}
\hat{J} = {} & \{\hat{M}^{(2)}+\lambda_2^2 c^{*(2)}\left(\lambda_1\right)-\frac{2\lambda_2^2 c^{*(1)}(\lambda_1)(\bar{v}+\bar{s}_1)}{1-\lambda_1\bar{b}_1}+\frac{2\lambda_2^2\bar{s}_1\left(c^{*(1)}(\lambda_1)v^*(\lambda_1)+c^*(\lambda_1)v^{*(1)}(\lambda_1)\right)}{1-\lambda_1\bar{b}_1}+ \\
& \frac{\lambda_2^2 c^*(\lambda_1)}{(1-\lambda_1\bar{b}_1)^2}[\bar{s}_1^{(2)}\lambda_1\sigma_{v^*}+2\bar{s}_1\bar{v}+\bar{v}^{(2)}+\frac{\lambda_1\bar{b}_1^{(2)}}{1-\lambda_1\bar{b}_1}\left(\bar{v}+\bar{s}_1\lambda_1\sigma_{v^*}\right)]\}/\left[c^*\left(\lambda_1\right)v^*\left(\lambda_1\right)\right]
\end{aligned}$$

$$\begin{aligned}
\hat{T}^{(2)} = {} & -\hat{K}^{(2)}c^*\left(\lambda_1\right)+2\lambda_2 c^{*(1)}\left(\lambda_1\right)\hat{K}^{(1)}-\lambda_2^2 c^{*(2)}\left(\lambda_1\right)\lambda_1\sigma_{s_2^*}-\hat{M}^{(2)}, \\
\hat{U}^{(2)} = {} & \frac{\hat{K}^{(2)}-2\hat{T}^{(1)}\hat{U}^{(1)}-\hat{T}^{(2)}U}{c^*(\lambda_1)s_2^*(\lambda_1)}.
\end{aligned}$$

Let now for any function $w\left(0,z_1,z_2\right)$ denote $\tilde{w}^{(\nu)}=\frac{d^\nu}{dz_2^\nu}\,w\left(0,1,z_2\right)\big|_{z_2=1}$. Then

$$\begin{aligned}
\tilde{K}^{(1)} &= \lambda_1\left[\lambda_2\bar{s}_1\sigma_{s_2^*}+\sigma_{s_2^*}^{(1)}\right], \quad \tilde{K}^{(2)}=\lambda_1\left[\sigma_{s_2^*}^{(2)}+\lambda_2^2\bar{s}_1^{(2)}\sigma_{s_2^*}+2\lambda_2\bar{s}_1\sigma_{s_2^*}^{(1)}\right], \\
\tilde{M}^{(1)} &= \lambda_1\left[\lambda_2\bar{s}_3\sigma_{c^*}+\sigma_{c^*}^{(1)}\right], \quad \tilde{M}^{(2)}=\lambda_1\left[\sigma_{c^*}^{(2)}+\lambda_2^2\bar{s}_3^{(2)}\sigma_{c^*}+2\lambda_2\bar{s}_3\sigma_{c^*}^{(1)}\right], \\
\tilde{T}^{(1)} &= -\tilde{K}^{(1)}c^*\left(\lambda_1\right)+\lambda_2 c^{*(1)}\left(\lambda_1\right)\lambda_1\sigma_{s_2^*}-\tilde{M}^{(1)}, \\
\tilde{T}^{(2)} &= -\tilde{K}^{(2)}c^*\left(\lambda_1\right)+2\lambda_2 c^{*(1)}\left(\lambda_1\right)\tilde{K}^{(1)}-\lambda_2^2 c^{*(2)}\left(\lambda_1\right)\lambda_1\sigma_{s_2^*}-\tilde{M}^{(2)}, \\
\vec{R}^{(1)} &= -\frac{d}{dz_2}\vec{R}(0,1,z_2)|_{z_2=1}=\tilde{T}^{(1)}U+c^*(\lambda_1)s_2^*(\lambda_1)\hat{U}^{(1)}-\tilde{K}^{(1)}, \\
\vec{R}^{(2)} &= -\frac{d^2}{dz_2^2}\vec{R}(0,1,z_2)|_{z_2=1}=\tilde{T}^{(2)}U+2\tilde{T}^{(1)}\hat{U}^{(1)}+c^*(\lambda_1)s_2^*(\lambda_1)\hat{U}^{(2)}-\tilde{K}^{(2)},
\end{aligned}$$

and finally

$$E\left((N_2;\xi=b_1,u=1)\right)=\frac{H_1}{\lambda_2}\vec{R}^{(1)}+\frac{\vec{R}^{(2)}}{2}.$$

For the computation of $E\left((N_2;\xi=b_1,u=0)\right)$ we need finally the derivatives, at the point $z_2=1$, of the functions $L(0,1,z_2)$, $h_i(0,1,z_2)$, $i=1,2$, and $e_1(0,1,z_2)=1/e(0,1,z_2)$ defined in (8), (28) and (5) respectively. Thus

$$\begin{aligned}
\tilde{L}^{(1)} = &\ -\lambda_2 s_2^{*(1)}(\lambda_1)+\lambda_2 s_2^*(\lambda_1)(\bar{b}_2+\bar{s}_1\lambda_1\sigma_{b_2^*}),\\
\tilde{L}^{(2)} = &\ \lambda_2^2 s_2^{*(2)}(\lambda_1)+2\lambda_2^2\bar{s}_1(s_2^{*(1)}(\lambda_1)\, b_2^*(\lambda_1)+s_2^*(\lambda_1)\, b_2^{*(1)}(\lambda_1))\\
&+\lambda_2^2 s_2^*(\lambda_1)(\bar{s}_1^{(2)}\lambda_1\sigma_{b_2^*}+2\bar{s}_1\bar{b}_2+\bar{b}_2^{(2)})-2\lambda_2^2 s_2^{*(1)}(\lambda_1)(\bar{b}_2+\bar{s}_1),\\
R^{(2)} = &\ \tfrac{d^2}{dz_2^2}R(0,z_2)|_{z_2=1}=\lambda_2^2 c^{*(2)}(\lambda_1)\, U-2\lambda_2 c^{*(1)}(\lambda_1)\,\hat{U}^{(1)}+c^*(\lambda_1)\,\hat{U}^{(2)},\\
\tilde{e}_1^{(1)} = &\ -\tfrac{\lambda_2}{\lambda_1}\left[\tfrac{\lambda_1\sigma_{v^*}(1+\lambda_1\bar{s}_3)+\lambda_1 c^*(\lambda_1)(\bar{v}+\bar{s}_1\lambda_1\sigma_{v^*})}{c^*(\lambda_1)v^*(\lambda_1)}\right],\\
\tilde{e}_1^{(2)} = &\ 2\lambda_2\tfrac{\left[c^{*(1)}(\lambda_1)v^*(\lambda_1)+c^*(\lambda_1)v^{*(1)}(\lambda_1)\right]}{c^*(\lambda_1)v^*(\lambda_1)}\tilde{e}_1^{(1)}-(1-\lambda_1\bar{b}_1)\hat{J}+\tfrac{\lambda_1\lambda_2^2}{(1-\lambda_1\bar{b}_1)^2}\\
&\times[\tfrac{\bar{b}_1(1-\lambda_1\bar{b}_1)(\sigma_{c^*}(\lambda_1\bar{s}_3^{(2)}+2\bar{s}_3)+c^*(\lambda_1)(\bar{s}_1^{(2)}\lambda_1\sigma_{v^*}+2\bar{s}_1\bar{v}+\bar{v}^{(2)})}{c^*(\lambda_1)v^*(\lambda_1)}\\
&+\tfrac{\bar{b}_1^{(2)}(\sigma_{c^*}(1+\lambda_1\bar{s}_3)+c^*(\lambda_1)(\bar{v}+\lambda_1\sigma_{v^*}\bar{s}_1))}{c^*(\lambda_1)v^*(\lambda_1)}]
\end{aligned}$$

and

$$\begin{aligned}
\tilde{h}_1^{(2)} &= a\{\tilde{L}^{(2)}/s_2^*(\lambda_1)+2\tilde{L}^{(1)}R^{(1)}+s_2^*(\lambda_1)\tilde{R}^{(2)}-2\tilde{e}_1^{(1)}-\tilde{e}_1^{(2)}\},\\
\tilde{h}_2^{(2)} &= \lambda_1\lambda_2^2\bar{s}_1^{(2)}-\lambda_1\tilde{e}_1^{(2)}+\tfrac{\lambda_2}{a}\tilde{h}_1^{(2)},\\
\tilde{h}_1^{(1)} &= a\{\tilde{L}^{(1)}/s_2^*(\lambda_1)+s_2^*(\lambda_1)R^{(1)}-1-\tilde{e}_1^{(1)}\},\\
\tilde{h}_2^{(1)} &= \lambda_1\lambda_2\bar{s}_1+\tfrac{\lambda_2}{a}\tilde{h}_1^{(1)}-\lambda_1\tilde{e}_1^{(1)}.
\end{aligned}$$

Then finally

$$\begin{aligned}
E\left(N_2;\xi=b_1,u=0\right) = &\ -\tfrac{1}{2\lambda_2}\{2\tilde{h}_1^{(1)}\tfrac{1-\lambda_1\bar{b}_1}{2a(1-\rho)}\left[\lambda_2\rho_w^{(2)}+\lambda_1\rho_d^{(2)}Q^*(1)\right]+\\
&\ \tilde{h}_2^{(2)}Q^*(1)+\left(\tilde{h}_1^{(2)}+2\tilde{h}_2^{(1)}\right)\tfrac{Q^*(1)}{a}\left(\tfrac{\lambda_1\bar{\rho}_d+\lambda_2(\bar{\rho}_c+\bar{\rho}_v)}{1-\rho}\right)\},
\end{aligned}$$

with

$$\rho_r=\frac{\lambda_2\sigma_{c^*}(1+\lambda_1\bar{s}_3)}{(1-\lambda_1\bar{b}_1)\, c^*(\lambda_1)},$$

$$\begin{aligned}
\rho_r^{(2)} = &\ \tfrac{2\lambda_2^2[c^{*(1)}(\lambda_1)(1+\lambda_1\bar{s}_3)+c^*(\lambda_1)\sigma_{c^*}]}{\lambda_1(1-\lambda_1\bar{b}_1)(c^*(\lambda_1))^2}+\tfrac{\lambda_2^2\sigma_{c^*}(\lambda_1\bar{s}_3^{(2)}+2\bar{s}_3)}{c^*(\lambda_1)(1-\lambda_1\bar{b}_1)^2}+2\rho_r^2+\tfrac{\lambda_1\lambda_2\bar{b}_1^{(2)}\rho_r}{(1-\lambda_1\bar{b}_1)^2},\\
\rho_d^{(2)} = &\ \tfrac{\lambda_2^2\bar{b}_1^{(2)}}{(1-\lambda_1\bar{b}_1)^3}(1+\lambda_1\bar{s}_1)+2\tfrac{\lambda_2(\bar{b}_1+\bar{s}_1)}{(1-\lambda_1\bar{b}_1)}\hat{e}^{(1)}+\tfrac{\lambda_2^2}{(1-\lambda_1\bar{b}_1)^2}(\bar{s}_1^{(2)}+2\bar{b}_1\bar{s}_1)\ +\hat{e}^{(2)},
\end{aligned} \tag{44}$$

$$\begin{aligned}
\rho_w^{(2)} = &\ \tfrac{1}{s_2^*(\lambda_1)}\{\ \hat{L}^{(2)}+\hat{K}^{(2)}+2(\hat{L}^{(1)}\hat{e}^{(1)}+\hat{K}^{(1)}\rho_r)+s_2^*(\lambda_1)\,\hat{e}^{(2)}+\lambda_1\sigma_{s_2^*}\rho_r^{(2)}\\
&+2\lambda_2\rho_w\{s_2^{*(1)}(\lambda_1)+\tfrac{\sigma_{s_2^*}[\lambda_1\sigma_{c^*}(1+\lambda_1\bar{s}_3)+c^*(\lambda_1)(1+\lambda_1\bar{s}_1)]}{(1-\lambda_1\bar{b}_1)c^*(\lambda_1)}\ \}\}.
\end{aligned} \tag{45}$$

8 Conclusions

In this paper a queuing model with two kind of customers, ordinary and retrial customers, is studied. To start serving both type of customers, the server needs a start up time, while when there are no customers waiting service, the server performs a close down period and in the sequel he departs for a single vacation. Upon discovering a Markov Renewal Process at particular time epochs, we describe our system as a Semi Regenerating Process and use the theory of Markov Renewal Processes to derive conditions for the system stability. Moreover, using the supplementary variable technique, we obtain expressions for the generating functions of the system state probabilities, both in a transient and in a steady state, and use them to derive expressions for the mean number of customers in the system, and the proportion of time the server remains in a particular stage (idle, busy, in start up, in close down, in vacation). Although the model is quite general containing a large number of arbitrarily distributed random variables, the obtained expressions are easily computable and can be directly used to produce numerical results and to compare system performance, under different values of the parameters.

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